MATH 1650: SECTION 2.2: FACTORING AND DIVISION

RECALL: The **zeros** of a function f are the solutions to the equation $f(x) = 0$.

NOTE: Geometrically, c is a zero of f means $(c, 0)$ is an x-intercept of the graph of $y = f(x)$.

In the previous section, the polynomial functions were already factored or partially factored so finding the zeros wasn't too involved. In this section, we explore what happens if the polynomials aren't factored so nicely for us.

We first take a minute to review some helpful strategies from intermediate algebra.

STRATEGIES FROM INTERMEDIATE ALGEBRA:

- Does the polynomial factor by grouping?
- Is the polynomial a 'quadratic in disguise'? That is:
  - The polynomial has exactly three terms.
  - The exponent on one term is exactly twice the exponent on the other.

EXAMPLE: Find the real zeros of the following functions:

- $p(x) = 2x^3 - 10x^2 - 3x + 15$

  We set $p(x) = 2x^3 - 10x^2 - 3x + 15 = 0$. Since $p(x)$ has four terms, we try factoring by grouping.

  We try factoring out the G.C.F. from **pairs** of terms, and see if this reveals a common factor.

  If we group the first two terms, we can factor out a $2x^2$ to get $2x^3 - 10x^2 = 2x^2(x - 5)$.

  We now try to factor something out of the last two terms that will leave us with a factor of $(x - 5)$.

  Sure enough, we can factor out a -3 from both: $-3x + 15 = -3(x - 5)$.

  Hence, we get: $2x^3 - 10x^2 - 3x + 15 = 2x^2(x - 5) - 3(x - 5) = (2x^2 - 3)(x - 5)$.

  Back to solving $2x^3 - 10x^2 - 3x + 15 = 0$. Factoring, we get $(2x^2 - 3)(x - 5) = 0$.

  Hence, $2x^2 - 3 = 0$ or $x - 5 = 0$. We get $x = \pm\sqrt{\dfrac{3}{2}} = \pm\dfrac{\sqrt{6}}{2}$ or $x = 5$.

- $q(x) = x^4 - x^2 - 12$

  We set $q(x) = x^4 - x^2 - 12 = 0$. We notice:

  - there are three terms
  - and the exponent on one term (4) is twice the exponent on another term (2).

  Hence, we treat this as a quadratic. We find: $x^4 - x^2 - 12 = (x^2 - 4)(x^2 + 3)$.

  Back to solving $q(x) = x^4 - x^2 - 12 = 0$. We now have $(x^2 - 4)(x^2 + 3) = 0$.

  Solving $x^2 - 4 = 0$ gives $x = \pm 2$ while $x^2 + 3 = 0$ gives $x^2 = -3$ which has no real solutions.

  Hence, $x = \pm 2$ are the only real zeros of q.

EXAMPLE: Find all real zeros of the given polynomial. Check your answer using a graphing utility.

- $p(x) = 2x^3 - x^2 - 20x + 10$

- $f(t) = t^4 - 8t^2 + 15$

SYNTHETIC DIVISION AND THE FACTOR THEOREM

If the polynomial function doesn't factor using previously known techniques, we need to attack the problem the 'old fashioned' way an try to factor the polynomial using division. Consider the problem: $(x^3+4x^2-5x-14)\div(x-2)$

Using Long Division, we get:

$$
\begin{array}{r}
x^2 \;+\; 6x \;+\; 7 \\
x-2\,\overline{)\,x^3 + 4x^2 - \;5x \;-\; 14} \\
-\underline{(x^3-2x^2)} \qquad\qquad\quad \\
6x^2 - \;5x \qquad\;\; \\
-\underline{(6x^2-12x)} \qquad\; \\
7x \;-\; 14 \\
-\;\underline{(7x\;-14)} \\
0
\end{array}
$$

Since we get a remainder of zero, we get $x^3+4x^2-5x-14=(x-2)(x^2+6x+7)$.

We can take advantage of polynomial long division to streamline this process. First off, let's change all of the subtractions into additions by distributing through the -1s.

$$
\begin{array}{r}
x^2 \;+\; 6x \;+7 \\
x-2\,\overline{)\,x^3 + 4x^2 - \;5x \;-14} \\
\underline{-x^3+\;2x^2} \qquad\qquad\;\; \\
6x^2 - \;5x \qquad\; \\
\underline{-6x^2+12x} \qquad\; \\
7x \;-14 \\
\underline{-7x+14} \\
0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\,\overline{)\,x^3+4x^2-\;5x\;-14} \\
\underline{\;\;2x^2} \qquad\qquad\quad \\
6x^2 \qquad\qquad\quad \\
\underline{\qquad 12x} \qquad\;\; \\
7x \qquad\;\; \\
\underline{\qquad 14} \\
0
\end{array}
$$

Let's move terms up a bit and copy the x^3 into the last row.

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\,\overline{)\,x^3+4x^2-\;5x\;-14} \\
\underline{\qquad 2x^2 \quad 12x \quad 14} \\
x^3 \quad 6x^2 \quad 7x \quad\; 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$\begin{array}{r|rrrr} -2 & x^3 & +4x^2 & -\ 5x & -14 \\ & & 2x^2 & 12x & 14 \\ \hline & x^3 & 6x^2 & 7x & 0 \end{array}$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

EXAMPLE: Perform the following problems using synthetic division. Identify the quotient and the remainder.

- $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$

  HINT: Since there is no 'x' term, record a '0' in the synthetic division tableu as a placeholder.

- $\left(t^3 + 8\right) \div (t + 2)$

  HINT: Record '0' in the synthetic division tableu as placeholders as needed. Also note, $t + 2 = t - (-2)$.

Let $p(x) = 5x^3 - 2x^2 + 1$. We know from above that the remainder when $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$ is 118.
This means $p(x) = 5x^3 - 2x^2 + 1 = (x - 3)\left(5x^2 + 13x + 39\right) + 118$.
Hence if we want to find $p(3)$, we can use the formula $p(x) = (x - 3)\left(5x^2 + 13x + 39\right) + 118$.
Doing so, we quickly find $p(3) = ((3) - 3)\left(5(3)^2 + 13(3) + 39\right) + 118 = 0(\text{something}) + 118 = 118$.
Hence, $p(3)$ is the same as the remainder when $p(x)$ is divided by $(x - 3)$.
This is not a coincidence and happens all the time. This result is called The Remainder Theorem.

THE REMAINDER THEOREM: Suppose p is a polynomial function of degree at least 1 and c is a real number. When $p(x)$ is divided by $(x - c)$ the remainder is $p(c)$.

Let $p(t) = t^3 + 8$. We know from above that the remainder when $\left(t^3 + 8\right) \div (t + 2)$ is 0.
The Remainder Theorem tells us then that $p(-2) = 0$. Moreover, we see that $p(t) = (t + 2)\left(t^2 - 2t + 4\right)$.
In other words, $p(-2) = 0$ means that $(t + 2)$ is a factor of $p(t)$. This is an example of the Factor Theorem.

THE FACTOR THEOREM: Suppose p is a nonzero polynomial function. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

The Factor Theorem tells us what we'd already suspected: to find the zeros of a polynomial function, we need to factor. Moreover, all zeros of polynomial functions come from factoring.
Our big takeaway: x-intercepts correspond to zeroes which correspond to factors. We can quickly identify factors using synthetic division and checking for a remainder of 0.

EXAMPLE: Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$:

- show $x = \frac{1}{2}$ is a zero of multiplicity 2

  To show $x = \frac{1}{2}$ is a zero of multiplicity 2, we need to show $\left(x - \frac{1}{2}\right)$ divides $p(x)$ **exactly twice**.

  We set up for synthetic division, but instead of stopping after the first division, we continue.

$$\begin{array}{c|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
 & 4 & 0 & -12 & \boxed{0} &
\end{array}$$

  We get: $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2 \left(4x^2 - 12\right)$.

  Hence, we know $\left(x - \frac{1}{2}\right)$ divides $p(x)$ **at least** twice, so $x = \frac{1}{2}$ is a zero of **at least** multiplicity 2.

  To show $\left(x - \frac{1}{2}\right)$ divides $p(x)$ **only** twice, we check to see if $\left(x - \frac{1}{2}\right)$ divides $\left(4x^2 - 12\right)$.

  Substituting $x = \frac{1}{2}$ into $4x^2 - 12$ gives $4\left(\frac{1}{2}\right)^2 - 12 = -11 \neq 0$.

  Hence, by the Factor Theorem, $\left(x - \frac{1}{2}\right)$ cannot possibly divide $\left(4x^2 - 12\right)$.

  Hence, $\left(x - \frac{1}{2}\right)$ divides $p(x)$ **exactly** twice, so $x = \frac{1}{2}$ is a zero of multiplicity 2.

- find all of the remaining real zeros of p

  To find the remaining zeros of p, we set the quotient, $4x^2 - 12 = 0$. We get $4x^2 = 12$ or $x^2 = 3$.

  Extracting square roots, we get $x = \pm\sqrt{3}$.

EXAMPLE: Let $p(x) = 2x^4 + 11x^3 + 11x^2 - 15x - 9$.

- Use the fact that $x = -3$ is a zero of multiplicity 2 to find the remaining real zeros of p.

HOMEWORK: Section 2.2: 1 - 35 odd